

Dimensions and spectral triples for fractals in \mathbb{R}^N

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Abstract

Two spectral triples are introduced for a class of fractals in \mathbb{R}^N .

The definitions of noncommutative Hausdorff dimension and noncommutative tangential dimensions, as well as the corresponding Hausdorff and Hausdorff-Besicovitch functionals considered in [7], are studied for the mentioned fractals endowed with these spectral triples, showing in many cases their correspondence with classical objects. In particular, for any limit fractal, the Hausdorff-Besicovitch functionals do not depend on the generalized limit ω .

0 Introduction.

In this paper we extend the analysis we made in [7] to fractals in \mathbb{R}^N , more precisely we define spectral triples for a class of fractals and compare the classical measures, dimensions and metrics with the measures, dimensions and metrics obtained from the spectral triple, in the framework of A. Connes’ noncommutative geometry [2].

The class of fractals we consider is the class of limit fractals, namely fractals which can be defined as Hausdorff limits of sequences of compact sets obtained via sequences of contraction maps. This class contains the self-similar fractals and is contained in the wider class of random fractals [12]. On any limit fractal, the described limit procedure produces also a family of limit measures μ_α , $\alpha > 0$. Among limit fractals, we consider in particular the translation fractals, namely those for which the generating similarities of a given level have the same similarity parameter. It turns out that for translation fractals all the limit measures μ_α coincide.

For limit fractals we introduce spectral triples which generalise the one considered by Connes in [2] for Cantor-like fractals, namely are based on an approximation of the fractal with sequences of pairs of points. In the first spectral triple, the sequences consist of all descendants, via the generating similarities,

of one (or finitely many) ancestral pair. In the second triple, among the descendants of a single ancestor via the generating similarities, we consider all parent-child pairs.

In both cases, when translation fractals are considered, we prove that the noncommutative Hausdorff dimension and tangential dimensions defined in [7] coincide with their classical counterparts computed in [9, 10]. Let us recall that the noncommutative tangential dimensions are the extreme points of the traceability interval, namely of the set of (singular) traceability exponents for the inverse modulus of the Dirac operator. Therefore any of these exponents gives rise to a singular trace τ_ω which in turn defines a trace on the algebra \mathcal{A} of the spectral triple, hence, by Riesz theorem, a measure on the fractal. For translation fractals all these measures coincide with the limit measure. In the case of the parent-child triple, an analogous result holds for any limit fractal, i.e. the measure coming from the traceability exponent α coincides with the limit measure μ_α . As a consequence, the measure generated by a singular trace τ_ω is well defined, namely does not depend on the generalised limit procedure ω .

Finally we study the distance on the fractal induced *à la Connes* by the spectral triple. In the case of the parent-child triple, the noncommutative distance is always equivalent to the Euclidean distance, namely they induce the same topology. Then we compare the noncommutative distance with the Euclidean geodesic distance, namely with the distance defined in terms of rectifiable curves contained in the fractal (when they exist). We prove that the identity map from the fractal endowed with the geodesic distance to the fractal endowed with the noncommutative distance is Lipschitz. As a consequence, when the Euclidean distance and the geodesic distance are bi-Lipschitz, this holds for the noncommutative distance too.

1 Classical aspects

We start this Section by recalling known results on self-similar fractals, then we introduce the class of limit fractals and their limit measures, and give some examples. We then introduce an open set condition which allows us to characterise the limit measures on the fractal (Theorem 1.7), and to compute them in case of translation limit fractals, under a mild assumption (Theorem 1.8). Finally, we recall the notions of tangential dimensions for metric spaces and measures from [9, 10].

1.1 Preliminaries

The general reference for this Subsection is [4].

Hausdorff measure and dimension. Let (X, ρ) be a metric space, and let $h : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing and right-continuous, with $h(0) = 0$. When $E \subset X$, define, for any $\delta > 0$, $\mathcal{H}_\delta^h(E) := \inf \{ \sum_{i=1}^\infty h(\text{diam } A_i) : \cup_i A_i \supset E, \text{diam } A_i \leq \delta \}$. Then the *Hausdorff-Besicovitch (outer) measure* of E

is defined as

$$\mathcal{H}^h(E) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(E).$$

If $h(t) = t^\alpha$, \mathcal{H}^α is called *Hausdorff (outer) measure* of order $\alpha > 0$.
The number

$$d_H(E) := \sup\{\alpha > 0 : \mathcal{H}^\alpha(E) = +\infty\} = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\}$$

is called *Hausdorff dimension* of E .

Selfsimilar fractals. Let $\{w_j\}_{j=1,\dots,p}$ be contracting similarities of \mathbb{R}^N , i.e. there are $\lambda_j \in (0, 1)$ such that $\|w_j(x) - w_j(y)\| = \lambda_j \|x - y\|$, $x, y \in \mathbb{R}^N$. Denote by $\mathcal{K}(\mathbb{R}^N)$ the family of all non-empty compact subsets of \mathbb{R}^N , endowed with the Hausdorff metric, which turns it into a complete metric space. Then $W : K \in \mathcal{K}(\mathbb{R}^N) \rightarrow \cup_{j=1}^p w_j(K) \in \mathcal{K}(\mathbb{R}^N)$ is a contraction.

Definition 1.1. The unique non-empty compact subset F of \mathbb{R}^N such that

$$F = W(F) = \bigcup_{j=1}^p w_j(F)$$

is called the *self-similar fractal* defined by $\{w_j\}_{j=1,\dots,p}$.

If we denote by $Prob_{\mathcal{K}}(\mathbb{R}^N)$ the set of probability measures on \mathbb{R}^N with compact support endowed with the Hutchinson metric, i.e. $d(\mu, \nu) := \sup\{|\int f d\mu - \int f d\nu| : \|f\|_{Lip} \leq 1\}$, then the map

$$\begin{aligned} T : Prob_{\mathcal{K}}(\mathbb{R}^N) &\rightarrow Prob_{\mathcal{K}}(\mathbb{R}^N) \\ \mu &\mapsto \sum_{j=1}^p \lambda_j^s \mu \circ w_j^{-1} \end{aligned}$$

is a contraction, where $s > 0$ is the unique real number, called similarity dimension, satisfying $\sum_{j=1}^p \lambda_j^s = 1$. We then observe that if μ has support K , then $T\mu$ has support $W(K)$. Since the sequence $W^n(K)$ is convergent, it turns out that it is bounded, namely there exists a compact set K_0 containing the supports of all the measures $T^n\mu$. But on the space $Prob(K_0)$ the Hutchinson metric induces the weak* topology, and this space is compact in such topology, hence complete in the Hutchinson metric. Therefore there exists a fixed point of T in $Prob_{\mathcal{K}}(\mathbb{R}^N)$, which is of course unique.

Open Set Condition. The similarities $\{w_j\}_{j=1,\dots,p}$ are said to satisfy the open set condition if there is a non-empty bounded open set $V \subset \mathbb{R}^N$ such that $\cup_{j=1}^p w_j(V) \subset V$ and $w_i(V) \cap w_j(V) = \emptyset$, $i \neq j$. In this case $d_H(F) = s$, the similarity dimension, and the Hausdorff measure \mathcal{H}^s is non-trivial on F . Therefore $\mathcal{H}^s|_F$ is the unique (up to a constant factor) Borel measure μ , with compact support, such that $\mu(A) = \sum_{j=1}^p \lambda_j^s \mu(w_j^{-1}(A))$, for any Borel subset A of \mathbb{R}^N .

1.2 Limit fractals.

Several generalisations of the class of self-similar fractals have been studied. Here we propose a new one, that we call the class of limit fractals. For its construction we need the following theorem.

Theorem 1.2. *Let (X, ρ) be a complete metric space, $T_n : X \rightarrow X$ be such that there are $\lambda_n \in (0, 1)$ for which $\rho(T_n x, T_n y) \leq \lambda_n \rho(x, y)$, for $x, y \in X$. Assume $\sum_{n=1}^{\infty} \prod_{j=1}^n \lambda_j < \infty$, and there is $x \in X$ such that $\sup_{n \in \mathbb{N}} \rho(T_n x, x) < \infty$. Then*

- (i) $\sup_{n \in \mathbb{N}} \rho(T_n y, y) < \infty$, for any $y \in X$,
- (ii) $\lim_{n \rightarrow \infty} T_1 \circ T_2 \circ \dots \circ T_n x = x_0 \in X$ for any $x \in X$.

Proof. (i) $\rho(T_n y, y) \leq \rho(T_n y, T_n x) + \rho(T_n x, x) + \rho(x, y) \leq (1 + \lambda_n) \rho(x, y) + \rho(T_n x, x)$, so that $\sup_{n \in \mathbb{N}} \rho(T_n y, y) \leq 2\rho(x, y) + \sup_{n \in \mathbb{N}} \rho(T_n x, x) < \infty$.

(ii) Set $M := \sup_{n \in \mathbb{N}} \rho(T_n x, x) < \infty$, and $S_n := T_1 \circ T_2 \circ \dots \circ T_n$, $n \in \mathbb{N}$. As $\rho(S_{n+1} x, S_n x) \leq \lambda_1 \lambda_2 \dots \lambda_n \rho(T_{n+1} x, x) \leq M \lambda_1 \lambda_2 \dots \lambda_n$, there follows, for any $p \in \mathbb{N}$, $\rho(S_{n+p} x, S_n x) \leq \rho(S_{n+p} x, S_{n+p-1} x) + \dots + \rho(S_{n+1} x, S_n x) \leq M \sum_{k=n}^{n+p-1} \prod_{j=1}^k \lambda_j \leq M \sum_{k=n}^{\infty} \prod_{j=1}^k \lambda_j \rightarrow 0$, as $n \rightarrow \infty$, that is $\{S_n x\}$ is Cauchy in X . Therefore there is $x_0 \in X$ such that $S_n x \rightarrow x_0$.

Let us prove that x_0 is independent of x . Indeed, if $y \in X$, then $\rho(S_n x, S_n y) \leq \lambda_1 \lambda_2 \dots \lambda_n \rho(x, y) \rightarrow 0$, as $n \rightarrow \infty$, so that $S_n x$ and $S_n y$ have the same limit. \square

Remark 1.3. A sufficient condition for $\sum_{n=1}^{\infty} \prod_{j=1}^n \lambda_j < \infty$ to hold is

$$\sup_{n \in \mathbb{N}} \lambda_n < 1.$$

We now describe the class of limit fractals. Let $\{w_{nj}\}$, $n \in \mathbb{N}$, $j = 1, \dots, p_n$, be contracting similarities of \mathbb{R}^N , with contraction parameter $\lambda_{nj} \in (0, 1)$. Set, for any $n \in \mathbb{N}$, $\Sigma_n := \{\sigma : \{1, \dots, n\} \rightarrow \mathbb{N} : \sigma(k) \in \{1, \dots, p_k\}, k = 1, \dots, n\}$, $\Sigma := \cup_{n \in \mathbb{N}} \Sigma_n$, $\Sigma_{\infty} := \{\sigma : \mathbb{N} \rightarrow \mathbb{N} : \sigma(k) \in \{1, \dots, p_k\}, k \in \mathbb{N}\}$, and write $w_{\sigma} := w_{1\sigma(1)} \circ w_{2\sigma(2)} \circ \dots \circ w_{n\sigma(n)}$, $\lambda_{\sigma} := \lambda_{1\sigma(1)} \lambda_{2\sigma(2)} \dots \lambda_{n\sigma(n)}$, for any $\sigma \in \Sigma_n$. Assume $\bar{\lambda} := \sup_{n,j} \lambda_{nj} < 1$ and $\{w_{\sigma}(x) : \sigma \in \Sigma\}$ is bounded, for some (hence any) $x \in \mathbb{R}^N$. Then, by Theorem 1.2, the sequence of maps $W_n : K \in \mathcal{K}(\mathbb{R}^N) \rightarrow \cup_{j=1}^{p_n} w_{nj}(K) \in \mathcal{K}(\mathbb{R}^N)$ is such that $\{W_1 \circ W_2 \circ \dots \circ W_n(K)\}$ has a limit in $\mathcal{K}(\mathbb{R}^N)$, which is independent of $K \in \mathcal{K}(\mathbb{R}^N)$.

Definition 1.4. The unique compact set F which is the limit of $\{W_1 \circ W_2 \circ \dots \circ W_n(K)\}_{n \in \mathbb{N}}$ is called the limit fractal defined by $\{w_{nj}\}$. In the particular case that $\lambda_{nj} = \lambda_n$, $j = 1, \dots, p_n$, $n \in \mathbb{N}$, F is called a translation (limit) fractal.

Example 1.5. As an example we mention some fractals considered in [11]. They are constructed as follows. At each step the sides of an equilateral triangle are divided in $q \in \mathbb{N}$ equal parts, so as to obtain q^2 equal equilateral triangles, and then all downward pointing triangles are removed, so that $\frac{q(q+1)}{2}$ triangles are left. The corresponding map W can therefore be described as the map which contracts the original triangle (or any of its subsets) by a factor $1/q$, and then puts a copy of it in each of the upward pointing triangles. Setting $q_j = 2$

if $(k-1)(2k-1) < j \leq (2k-1)k$ and $q_j = 3$ if $k(2k-1) < j \leq k(2k+1)$, $k = 1, 2, \dots$, we get a translation fractal with dimensions given by (see Theorem 1.14)

$$\underline{\delta} = \frac{\log 3}{\log 2} < \underline{d} = \bar{d} = \frac{\log 18}{\log 6} < \bar{\delta} = \frac{\log 6}{\log 3},$$

where $\underline{\delta}, \bar{\delta}, \underline{d}, \bar{d}$ denote the lower tangential, the upper tangential, the lower local and the upper local dimensions. The first four steps ($q = 2, 3, 3, 2$) of the procedure above are shown in Figure 1.

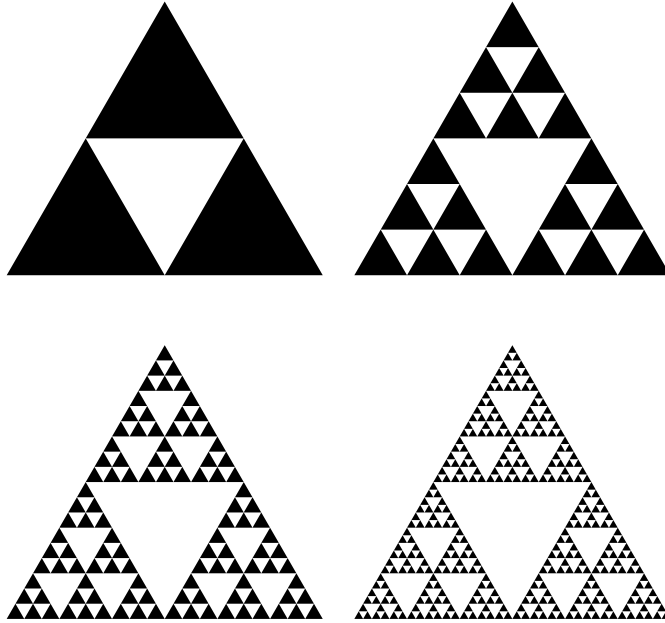


Figure 1: Modified Sierpinski

The procedure considered above can, of course, be applied also to other shapes. For example, at each step the sides of a square are divided in $2q+1$, $q \in \mathbb{N}$, equal parts, so as to obtain $(2q+1)^2$ equal squares, and then $2q(q+1)$ squares are removed, so that to remain with a chessboard. In particular, we may set $q_j = 2$ if $k(2k+1) < j \leq (2k+1)(k+1)$ and $q_j = 1$ if $k(2k-1) < j \leq k(2k+1)$, $k = 0, 1, 2, \dots$, getting a translation fractal with dimensions given by (see Theorem 1.14)

$$\underline{\delta} = \frac{\log 5}{\log 3} < \underline{d} = \bar{d} = \frac{\log 65}{\log 15} < \bar{\delta} = \frac{\log 13}{\log 5}.$$

The first three steps ($q = 1, 2, 1$) of this procedure are shown in Figure 2.

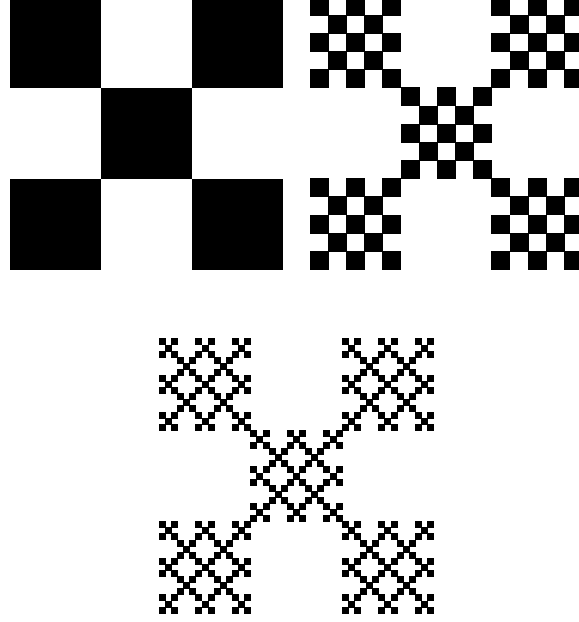


Figure 2: Modified Vicsek

As before, we may consider the action of the similarities on measures, besides that on sets. Given $\alpha > 0$ we set

$$T_n : \text{Prob}_{\mathcal{K}}(\mathbb{R}^N) \rightarrow \text{Prob}_{\mathcal{K}}(\mathbb{R}^N) \\ \mu \mapsto \frac{1}{\sum_{j=1}^p \lambda_{nj}^\alpha} \sum_{j=1}^p \lambda_{nj}^\alpha \mu \circ w_{nj}^{-1}$$

and consider the sequence $\{T_1 \circ T_2 \circ \dots \circ T_n \mu\}_{n \in \mathbb{N}}$. As before the supports of all such measures are contained in a common compact set, therefore Theorem 1.2 applies and we get a unique limit measure μ_α , depending on the chosen α .

As a consequence, if μ_0 is a probability measure and $\alpha > 0$, $\mu_n := T_1 \circ T_2 \circ \dots \circ T_n \mu_0$ satisfies

$$\mu_n(A) = \sum_{|\sigma|=n} c_{\sigma,\alpha} \mu_0(w_\sigma^{-1}(A)), \quad (1.1)$$

where

$$c_{\sigma,\alpha} := \frac{\lambda_\sigma^\alpha}{\sum_{|\sigma'|=|\sigma|} \lambda_{\sigma'}^\alpha}. \quad (1.2)$$

In particular, if F is a translation (limit) fractal,

$$c_{\sigma,\alpha} = \frac{1}{p_1 p_2 \dots p_{|\sigma|}}, \quad \forall \alpha > 0,$$

so that the limit measures μ_α all coincide, and will be denoted by μ_{lim} .

1.3 Hausdorff dimension and limit measures

Assumption 1.6. Open Set Condition: There is a bounded open set V such that $w_{ni}(V) \subset V$ for any $n \in \mathbb{N}$, $i \in \{1, \dots, p_n\}$ and $w_{n,i}V \cap w_{n,j}V = \emptyset$ if $i \neq j$. We also assume that V is regular, namely the Lebesgue measure of V is equal to the Lebesgue measure of its closure C and V is equal to the interior of C .

OSC implies that $w_{ni}C \subset C$ for any $n \in \mathbb{N}$, $i \in \{1, \dots, p_n\}$ and

$$S_{N+1}C = W_1 \cdots W_N \cdot W_{N+1}C \subset W_1 \cdots W_NC = S_NC,$$

namely $C_N = S_NC$ is a decreasing sequence of compact sets converging to F , $\cap C_N = F$. Now set

$$\begin{aligned} V_\sigma &= w_\sigma V, \\ C_\sigma &= w_\sigma C, \\ F_\sigma &= w_\sigma C \cap F = C_\sigma \cap F. \end{aligned}$$

Then

(i)

$$\bigcup_{|\sigma|=N} F_\sigma = \bigcup_{|\sigma|=N} w_\sigma C \cap F = S_NC \cap F = C_N \cap F = F,$$

(ii) if $\sigma \geq \sigma'$, namely σ' is a truncation of σ , then $V_\sigma \subset V_{\sigma'}$,

(iii) if σ, σ' are not ordered, $V_\sigma \cap V_{\sigma'} = \emptyset$.

Moreover, in this case, $C_\sigma \cap V_{\sigma'} = \emptyset$. In fact, if $x \in C_\sigma \cap V_{\sigma'}$, $x \in \partial V_\sigma$, hence there is a sequence $x_n \rightarrow x$, $x_n \in V_\sigma$, therefore x_n is eventually in $V_{\sigma'}$, against the hypothesis.

Theorem 1.7. Let F be a limit fractal satisfying OSC, with $\text{vol}(V) = \text{vol}(C)$. Then the limit measure μ_α is the unique probability measure satisfying the following property: for any subset \mathcal{J} of Σ_n

$$\mu_\alpha(V_{\mathcal{J}}) \leq \sum_{\sigma \in \mathcal{J}} c_{\sigma, \alpha} \leq \mu_\alpha(C_{\mathcal{J}}), \quad (1.3)$$

where we set $C_{\mathcal{J}} = \cup_{\sigma \in \mathcal{J}} C_\sigma$, $V_{\mathcal{J}}$ equal to the interior of $C_{\mathcal{J}}$ relative to C , and $c_{\sigma, \alpha}$ is defined in (1.2).

Proof. Let μ_n converging to μ_α be as described in (1.1). Since μ_α does not depend on the starting measure, we may set μ_0 as the normalized Lebesgue measure on V . Then μ_n eventually satisfies $\mu_n(V_\sigma) = \mu_n(C_\sigma) = c_{\sigma, \alpha}$, therefore μ_n converges as a sequence of functionals on the vector space generated by continuous functions and step functions constant on the V_σ 's, giving rise to a positive bounded functional $\tilde{\mu}$ on such space.

Then $\mu_\alpha(V_{\mathcal{J}})$ may be defined as the supremum of $\int f d\mu_\alpha$ with support of f contained in $V_{\mathcal{J}}$, hence is majorised by $\tilde{\mu}(V_{\mathcal{J}}) = \sum_{\sigma \in \mathcal{J}} c_{\sigma, \alpha}$. The second inequality of (1.3) is proved analogously. Now we show that these inequalities determine μ_α uniquely. Let μ be a probability measure satisfying (1.3). We observe that, for

any continuous function f , $\int f d\mu$ is well approximated by the lower Riemann sums with step functions constant on the V_σ 's, as soon as $|\sigma|$ is big enough, since $\text{diam}(V_\sigma) \leq \text{diam}(V)(\bar{\lambda})^{|\sigma|}$. Let us now consider the set $\{\min_{C_\sigma} f : \sigma \in \Sigma_n\}$ and denote its elements by f_1, \dots, f_k in increasing order. Then define \mathcal{I}_j as the set of $\sigma \in \Sigma_n$ such that $\min_{C_\sigma} f \geq f_j$, C_j as the union of the C_σ for $\sigma \in \mathcal{I}_j$, V_j as the interior of C_j . Then

$$\begin{aligned} f_1 + \sum_{i=2}^k (f_i - f_{i-1})\mu(V_i) &\leq f_1 + \sum_{i=2}^k (f_i - f_{i-1}) \left(\sum_{\sigma \in \mathcal{I}_i} c_{\sigma, \alpha} \right) \\ &\leq f_1 + \sum_{i=2}^k (f_i - f_{i-1})\mu(C_i) \leq \int f d\mu. \end{aligned}$$

The second term may be rewritten as

$$\sum_{\sigma \in \Sigma_n} \min_{C_\sigma} f c_{\sigma, \alpha},$$

therefore

$$\int f d\mu = \lim_{n \rightarrow \infty} \sum_{\sigma \in \Sigma_n} \min_{C_\sigma} f c_{\sigma, \alpha},$$

hence there is only one probability measure satisfying (1.3). \square

Let now F be a translation fractal ($\lambda_{n,i}$ independent of i), and, to avoid triviality, assume $p_n \geq 2$ for any $n \in \mathbb{N}$. The OSC condition implies $\text{vol}(S_{n-1}C) = \sum_{i=1}^{p_n} \text{vol}(w_{ni}S_{n-1}C) = p_n \lambda_n^N \text{vol}(S_{n-1}C)$, so that $2\lambda_n^N \leq 1$, i.e. $\lambda_n \leq 2^{-1/N}$. We set

$$\Lambda_n = \prod_{i=1}^n \lambda_i, \quad P_n = \prod_{i=1}^n p_i. \quad (1.4)$$

Theorem 1.8. *Let F be a translation fractal, with the notation above, and assume $p := \sup_n p_n < \infty$. Then*

$$d_H(F) = \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log 1/\Lambda_n}.$$

Moreover the Hausdorff measure corresponding to $d := d_H(F)$ is non trivial if and only if $\liminf (\log P_n - d \log 1/\Lambda_n)$ is finite.

Proof. Let us consider the family \mathcal{P} of finite coverings of F , the subfamily $\mathcal{P}(\Sigma)$ of coverings made from sets of $\{C_\sigma : \sigma \in \Sigma\}$, and the subfamily $\mathcal{P}'(\Sigma)$, whose coverings consist of C_σ , $\sigma \in \Sigma$, $|\sigma| = \text{const}$. If $P \in \mathcal{P}$, $|P|$ denotes the maximum diameter of the sets in P . Clearly, for any $\alpha > 0$, we have

$$\begin{aligned} \mathcal{H}_\alpha(F) &= \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}}} \sum_{E \in P} (\text{diam } E)^\alpha \leq \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha \\ &\leq \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}'(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha. \end{aligned}$$

We shall show that the last two terms are indeed equal, and that the second term is majorised by a constant times the first, from which we derive

$$\mathcal{H}_\alpha(F) \asymp \liminf_{n \rightarrow \infty} P_n \Lambda_n^\alpha,$$

hence the required equality and the last statement.

We may assume without restriction that the diameter of V is equal to one. Then set $a := \frac{\text{vol}(V)}{\text{vol}(B(0,2))}$, where vol denotes the Lebesgue measure. Then the number of disjoint copies of V intersecting a ball of radius 1 is not greater than the number of disjoint copies of V contained in a ball of radius 2 which is in turn lower equal than a^{-1} .

As a consequence, for any $x \in F$,

$$\#\{\sigma \in \Sigma_n : C_\sigma \cap B_F(x, \Lambda_n) \neq \emptyset\} \leq a^{-1}. \quad (1.5)$$

For any $\varepsilon > 0$, let $P = \bigcup_{i=1}^n E_i \in \mathcal{P}$, $\text{diam } E_i = r_i \leq \varepsilon$. Let now $x_i \in E_i$, $\Lambda_{n_i+1} \leq r_i \leq \Lambda_{n_i}$.

Since the set $I_{n_i} \subset \Sigma$ of multi-indices of length n_i such that $\bigcup_{\sigma \in I_{n_i}} C_\sigma \supset B(x_i, \Lambda_{n_i})$ has cardinality majorised by a^{-1} , and any such C_σ contains at most p elements $C_{\sigma'}$, with $|\sigma'| = n_i + 1$, then the set of multi-indices $I_{n_i+1} \subset \Sigma$ of length $n_i + 1$ such that $\bigcup_{\sigma \in I_{n_i+1}} C_\sigma \supset B_F(x_i, \Lambda_{n_i})$ has cardinality majorised by p/a . Therefore

$$\bigcup_{i=1}^n \bigcup_{\sigma \in I_{n_i+1}} C_\sigma$$

is a covering of F of diameter less than ε and

$$\sum_{i=1}^n \sum_{\sigma \in I_{n_i+1}} (\text{diam } C_\sigma)^\alpha \leq \sum_{i=1}^n \frac{p}{a} \Lambda_{n_i+1}^\alpha \leq \frac{p}{a} \sum_{i=1}^n r_i^\alpha. \quad (1.6)$$

As a consequence

$$\lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha \leq \frac{p}{a} \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha = \frac{p}{a} \mathcal{H}^\alpha(F).$$

Now, for any $n_1 \leq n_0$, let P be the optimal covering of F made of C_σ 's, with $n_1 \leq |\sigma| \leq n_0$, namely minimizing $\sum_{\sigma \in P} (\text{diam } C_\sigma)^\alpha$, and choose $C_{\sigma_0} \in P$ with $|\sigma_0| = n_0$. This means that there is a C_σ , $|\sigma| = n_0 - 1$, which is optimally covered by some $C_{\sigma'}$'s of diameter Λ_{n_0} . Therefore this should be true for all other σ of length n_0 , namely the optimal covering is made of C_σ 's of the same size. This shows the equality

$$\lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha = \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}'(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha$$

hence concludes the proof. \square

Remark 1.9. Let F be a translation fractal, with the notation above, and assume $p := \sup_n p_n < \infty$. Let $G \subset F$ be closed. Then, with $\mathcal{P}(\Sigma)$ denoting the family of finite coverings of G made from sets in $\{V_\sigma : \sigma \in \Sigma\}$, for any $\alpha > 0$

$$\mathcal{H}^\alpha(G) \asymp \lim_{\varepsilon \rightarrow 0} \inf_{\substack{|P| \leq \varepsilon \\ P \in \mathcal{P}(\Sigma)}} \sum_{E \in P} (\text{diam } E)^\alpha.$$

It is clear that if the F_σ 's with $\sigma \in \Sigma_n$ are essentially disjoint w.r.t. μ_α , then $\mu_\alpha(F_\sigma) = c_{\sigma, \alpha}$. Now we will discuss some conditions implying the vanishing of μ_α on the intersections of the F_σ 's.

Theorem 1.10. *Let F be a translation fractal for which $p := \sup_n p_n < \infty$, G a closed subset of F s.t. $d_H(G) < d_H(F)$. Then, $\mu_{\text{lim}}(G) = 0$.*

As a consequence, if $d_H(w_{ni}C \cap w_{nj}C \cap F) < d_H(F)$, for any n , $i \neq j$, then $\mu_{\text{lim}}(F_\sigma) = \frac{1}{P_{|\sigma|}}$.

Proof. Let α be s.t. $d_H(G) < \alpha < d_H(F)$, and $\varepsilon > 0$. Then, from Theorem 1.8 and the Remark following it, there is $n_0 \in \mathbb{N}$, s.t. $P_n \Lambda_n^\alpha \geq 1$, for all $n \geq n_0$, and there is $\mathcal{J} \subset \Sigma$ s.t. $|\sigma| \geq n_0$, for all $\sigma \in \mathcal{J}$, and $\sum_{\sigma \in \mathcal{J}} \Lambda_\sigma^\alpha \leq \varepsilon$, and G is contained in the interior of $\cup_{\sigma \in \mathcal{J}} F_\sigma$. By Urysohn's lemma, there is $f \in \mathcal{C}(F)$, $0 \leq f \leq 1$, $f(x) = 1$, $x \in G$, $\text{supp } f \subset \cup_{\sigma \in \mathcal{J}} F_\sigma$. Then, with μ_k as in (1.1), and μ_0 the normalised Lebesgue measure,

$$\begin{aligned} \mu_{\text{lim}}(G) &\leq \int f d\mu = \lim_{k \rightarrow \infty} \int f d\mu_k \leq \lim_{k \rightarrow \infty} \mu_k(\cup_{\sigma \in \mathcal{J}} F_\sigma) \\ &\leq \lim_{k \rightarrow \infty} \sum_{\sigma \in \mathcal{J}} \mu_k(F_\sigma) = \sum_{\sigma \in \mathcal{J}} \frac{1}{P_{|\sigma|}} = \sum_{\sigma \in \mathcal{J}} \frac{1}{P_{|\sigma|} \Lambda_{|\sigma|}^\alpha} \Lambda_{|\sigma|}^\alpha \\ &\leq \sum_{\sigma \in \mathcal{J}} \Lambda_{|\sigma|}^\alpha \leq \varepsilon. \end{aligned}$$

The thesis follows. □

1.4 Tangential dimensions

Let (X, d) be a metric space, $E \subset X$. Let us denote by $n(r, E) \equiv n_r(E)$, resp. $\bar{n}(r, E) \equiv \bar{n}_r(E)$, the minimum number of open, resp. closed, balls of radius r necessary to cover E , and by $\nu(r, E) \equiv \nu_r(E)$ the maximum number of disjoint open balls of E of radius r contained in E .

Definition 1.11. [9] Let (X, d) be a metric space, $E \subset X$, $x \in E$. We call upper, resp. lower tangential dimension of E at x the (possibly infinite) numbers

$$\begin{aligned} \underline{\delta}_E(x) &:= \liminf_{\lambda \rightarrow 0} \liminf_{r \rightarrow 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda}, \\ \overline{\delta}_E(x) &:= \limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log n(\lambda r, E \cap \overline{B}(x, r))}{\log 1/\lambda}. \end{aligned}$$

Tangential dimensions are invariant under bi-Lipschitz maps, and satisfy properties which are characteristic of a dimension function.

Let μ be a locally finite Borel measure, namely μ is finite on bounded sets.

Let us recall that the local dimensions of a measure at x are defined as

$$\underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$

$$\overline{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Now we introduce tangential dimensions for μ .

Definition 1.12. [10] The lower and upper tangential dimensions of μ are defined as

$$\underline{\delta}_\mu(x) := \liminf_{\lambda \rightarrow 0} \liminf_{r \rightarrow 0} \frac{\log \left(\frac{\mu(B(x, r))}{\mu(B(x, \lambda r))} \right)}{\log 1/\lambda} \in [0, \infty],$$

$$\overline{\delta}_\mu(x) := \limsup_{\lambda \rightarrow 0} \limsup_{r \rightarrow 0} \frac{\log \left(\frac{\mu(B(x, r))}{\mu(B(x, \lambda r))} \right)}{\log 1/\lambda} \in [0, \infty].$$

Theorem 1.13. [10] *Let μ be a locally finite Borel measure on X . Then the following holds.*

$$\underline{\delta}_\mu(x) \leq \underline{d}_\mu(x) \leq \overline{d}_\mu(x) \leq \overline{\delta}_\mu(x).$$

Tangential dimensions are invariant under bi-Lipschitz maps.

Theorem 1.14. [10] *Let F be a translation fractal with the notations above, $\mu = \mu_{\text{lim}}$, and assume $p := \sup_n p_n < \infty$. Then*

$$\underline{d}_\mu(x) = d_H(F) = \liminf_{n \rightarrow \infty} \frac{\log P_n}{\log 1/\Lambda_n},$$

$$\overline{d}_\mu(x) = \limsup_{n \rightarrow \infty} \frac{\log P_n}{\log 1/\Lambda_n},$$

$$\underline{\delta}_F(x) = \underline{\delta}_\mu(x) = \liminf_{n, k \rightarrow \infty} \frac{\log P_{n+k} - \log P_n}{\log 1/\Lambda_{n+k} - \log 1/\Lambda_n},$$

$$\overline{\delta}_F(x) = \overline{\delta}_\mu(x) = \limsup_{n, k \rightarrow \infty} \frac{\log P_{n+k} - \log P_n}{\log 1/\Lambda_{n+k} - \log 1/\Lambda_n}.$$

2 Noncommutative aspects

2.1 Singular traces on the compact operators of a Hilbert space.

In this section we recall the theory of singular traces on $\mathcal{B}(\mathcal{H})$ as it was developed by Dixmier [3], who first showed their existence, and then in [13], [1] and [5].

A singular trace on $\mathcal{B}(\mathcal{H})$ is a tracial weight vanishing on the finite rank projections. Any tracial weight is finite on an ideal contained in $\mathcal{K}(\mathcal{H})$ and may be decomposed as a sum of a singular trace and a multiple of the normal trace. Therefore the study of (non-normal) traces on $\mathcal{B}(\mathcal{H})$ is the same as the study of singular traces. Moreover, making use of unitary invariance, a singular trace of a given operator should depend only on its eigenvalue asymptotics, namely, if A and B are positive compact operators on \mathcal{H} and $\mu_n(A) = \mu_n(B) + o(\mu_n(B))$, μ_n denoting the n -th eigenvalue, then $\tau(A) = \tau(B)$ for any singular trace τ . The main problem about singular traces is therefore to detect which asymptotics may be “resummed” by a suitable singular trace, that is to say, which operators are singularly traceable.

In order to state the most general result in this respect we need some notation. Let A be a compact operator. Then we denote by $\{\mu_n(A)\}$ the sequence of the eigenvalues of $|A|$, arranged in non-increasing order and counted with multiplicity. We consider also the (integral) sequence $\{S_n(A)\}$ defined as follows:

$$S_n(A) = \begin{cases} S_n^\uparrow(A) := \sum_{k=1}^n \mu_k(A) & A \notin \mathcal{L}^1 \\ S_n^\downarrow(A) := \sum_{k=n+1}^\infty \mu_k(A) & A \in \mathcal{L}^1, \end{cases}$$

where \mathcal{L}^1 denotes the ideal of trace-class operators. We call a compact operator *singularly traceable* if there exists a singular trace which is finite non-zero on $|A|$. We observe that the domain of such singular trace should necessarily contain the ideal $\mathcal{I}(A)$ generated by A . A compact operator is called *eccentric* if

$$\frac{S_{2n_k}(A)}{S_{n_k}(A)} \rightarrow 1 \quad (2.1)$$

for a suitable subsequence n_k . Then the following theorem holds.

Theorem 2.1. [1] *A positive compact operator A is singularly traceable iff it is eccentric. In this case there exists a sequence n_k such that both condition (2.1) is satisfied and, for any generalised limit Lim_ω on ℓ^∞ , the positive functional*

$$\tau_\omega(B) = \begin{cases} \text{Lim}_\omega \left(\left\{ \frac{S_{n_k}(B)}{S_{n_k}(A)} \right\} \right) & B \in \mathcal{I}(A)_+ \\ +\infty & B \notin \mathcal{I}(A), B > 0, \end{cases}$$

is a singular trace whose domain is the ideal $\mathcal{I}(A)$ generated by A .

The best known eigenvalue asymptotics giving rise to a singular trace is $\mu_n \sim \frac{1}{n}$, which implies $S_n \sim \log n$. The corresponding logarithmic singular trace is generally called Dixmier trace.

Definition 2.2. [7] If A is a compact operator, set $f(t) = -\log \mu_A(e^t)$, $t \in \mathbb{R}$, where μ_A is the extension of $\mu_n(A)$ to a piecewise constant right continuous

function on $[0, \infty)$. Then define

$$\begin{aligned}\underline{\delta}(A) &:= \left(\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\log \frac{\mu_n(A)}{\mu_{kn}(A)}}{\log k} \right)^{-1} = \left(\lim_{h \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{f(t+h) - f(t)}{h} \right)^{-1} \\ \bar{\delta}(A) &:= \left(\lim_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{\log \frac{\mu_n(A)}{\mu_{kn}(A)}}{\log k} \right)^{-1} = \left(\lim_{h \rightarrow \infty} \liminf_{t \rightarrow \infty} \frac{f(t+h) - f(t)}{h} \right)^{-1} \\ \bar{d}(A) &:= \left(\liminf_{n \rightarrow \infty} \frac{\log \mu_n(A)}{\log 1/n} \right)^{-1} = \left(\liminf_{t \rightarrow \infty} \frac{f(t)}{t} \right)^{-1}.\end{aligned}$$

Moreover, we say that $\alpha > 0$ is an exponent of singular traceability for A if $|A|^\alpha$ is singularly traceable.

Theorem 2.3. [7] *Let A be a compact operator. Then, the set of singular traceability exponents of A is the relatively closed interval in $(0, \infty)$ whose endpoints are $\underline{\delta}(A)$ and $\bar{\delta}(A)$. In particular, if $\bar{d}(A)$ is finite nonzero, it is an exponent of singular traceability.*

Note that the interval of singular traceability may be $(0, \infty)$, as shown in [6]. In [8] the previous Theorem has been generalised to any semifinite factor, and some questions concerning the domain of a singular trace have been considered.

2.2 Singular traces and spectral triples

In this section we shall discuss some notions of dimension in noncommutative geometry in the spirit of Hausdorff-Besicovitch theory.

As is known, the measure for a noncommutative manifold is defined via a singular trace applied to a suitable power of some geometric operator (e.g. the Dirac operator of the spectral triple of Alain Connes). Connes showed that such procedure recovers the usual volume in the case of compact Riemannian manifolds, and more generally the Hausdorff measure in some interesting examples [2], Section IV.3.

Let us recall that $(\mathcal{A}, \mathcal{H}, D)$ is called a *spectral triple* when \mathcal{A} is an algebra acting on the Hilbert space \mathcal{H} , D is a self adjoint operator on the same Hilbert space such that $[D, a]$ is bounded for any $a \in \mathcal{A}$, and D has compact resolvent. In the following we shall assume that 0 is not an eigenvalue of D , the general case being recovered by replacing D with $D|_{\ker(D)^\perp}$. Such a triple is called d^+ -summable, $d \in (0, \infty)$, when $|D|^{-d}$ belongs to the Macaev ideal $\mathcal{L}^{1,\infty} = \{a : \frac{S_n^\dagger(a)}{\log n} < \infty\}$.

The noncommutative version of the integral on functions is given by the formula $\text{Tr}_\omega(a|D|^{-d})$, where Tr_ω is the Dixmier trace, i.e. a singular trace summing logarithmic divergences. By the arguments below, such integral can be non-trivial only if d is the Hausdorff dimension of the spectral triple, but even this choice does not guarantee non-triviality. However, if d is finite non-zero, we may always find a singular trace giving rise to a non-trivial integral.

Theorem 2.4. [7] *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. If s is an exponent of singular traceability for $|D|^{-1}$, namely there is a singular trace τ which is non-trivial on the ideal generated by $|D|^{-s}$, then the functional $a \mapsto \tau(a|D|^{-s})$ is a trace state (Hausdorff-Besicovitch functional) on the algebra \mathcal{A} .*

Remark 2.5. When $(\mathcal{A}, \mathcal{H}, D)$ is associated to an n -dimensional compact manifold M , or to the fractal sets considered in [2], the singular trace is the Dixmier trace, and the associated functional corresponds to the Hausdorff measure. This fact, together with the previous theorem, motivates the following definition.

Definition 2.6. [7] Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, Tr_ω the Dixmier trace.

- (i) We call α -dimensional Hausdorff functional the map $a \mapsto \text{Tr}_\omega(a|D|^{-\alpha})$;
- (ii) we call (Hausdorff) dimension of the spectral triple the number

$$d(\mathcal{A}, \mathcal{H}, D) = \inf\{d > 0 : |D|^{-d} \in \mathcal{L}_0^{1,\infty}\} = \sup\{d > 0 : |D|^{-d} \notin \mathcal{L}^{1,\infty}\},$$

where $\mathcal{L}_0^{1,\infty} = \{a : \frac{S_n^\dagger(a)}{\log n} \rightarrow 0\}$.

(iii) we call minimal, resp. maximal dimension of the spectral triple the quantity $\underline{\delta}(|D|^{-1})$, resp. $\overline{\delta}(|D|^{-1})$, hence $|D|^\alpha$ is singularly traceable iff $\alpha \in [\underline{\delta}, \overline{\delta}] \cap (0, +\infty)$.

(iv) For any s between the minimal and the maximal dimension, we call the corresponding trace state on the algebra \mathcal{A} a Hausdorff-Besicovitch functional on $(\mathcal{A}, \mathcal{H}, D)$.

Theorem 2.7. [7]

- (i) $d(\mathcal{A}, \mathcal{H}, D) = \overline{\delta}(|D|^{-1})$.
- (ii) $d := d(\mathcal{A}, \mathcal{H}, D)$ is the unique exponent, if any, such that the d -dimensional Hausdorff functional is non-trivial.
- (iii) If $d \in (0, \infty)$, it is an exponent of singular traceability.

Let us observe that a singular trace, hence in particular the α -dimensional Hausdorff functional, depends on a generalized limit procedure ω , however its value is uniquely determined on the operators $a \in \mathcal{A}$ such that $a|D|^{-d}$ is measurable in the sense of Connes [2]. By an abuse of language we call measurable such operators.

As in the commutative case, the dimension is the supremum of the α 's such that the α -dimensional Hausdorff measure is everywhere infinite and the infimum of the α 's such that the α -dimensional Hausdorff measure is identically zero. Concerning the non-triviality of the d -dimensional Hausdorff functional, we have the same situation as in the classical case. Indeed, according to the previous result, a non-trivial Hausdorff functional is unique (on measurable operators) but does not necessarily exist. In fact, if the eigenvalue asymptotics of D is e.g. $n \log n$, the Hausdorff dimension is one, but the 1-dimensional Hausdorff measure gives the null functional.

However, if we consider all singular traces, not only the logarithmic ones, and the corresponding trace functionals on \mathcal{A} , as we said, there exists a non trivial trace functional associated with $d(\mathcal{A}, \mathcal{H}, D) \in (0, \infty)$, but $d(\mathcal{A}, \mathcal{H}, D)$ is not characterized by this property. In fact this is true if and only if the minimal and the maximal dimension coincide. A sufficient condition is the following.

Proposition 2.8. [7] *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with finite non-zero dimension d . If there exists $\lim \frac{\mu_n(D^{-1})}{\mu_{2n}(D^{-1})} \in (1, \infty)$, d is the unique exponent of singular traceability of D^{-1} .*

2.3 A spectral triple for fractals

In this Subsection we introduce a spectral triple on limit fractals, by extending an idea of Connes for Cantor-like fractals. We compute its various dimensions, and recognise the noncommutative Hausdorff functional as the one arising from the limit measure on the fractal. Little can we say on the metric defined by this spectral triple, so in the next Subsection we propose a different spectral triple.

Let F be a limit fractal which satisfies OSC (se Assumption 1.6) with respect to the open set V , and let C be the closure of V . Choose two points $x, y \in C$, and denote with r their distance. Also, construct the sequences $x_\sigma = w_\sigma x$, $y_\sigma = w_\sigma y$, $\sigma \in \Sigma$ and note that

$$\|x_\sigma - y_\sigma\| = r_\sigma := r\lambda_\sigma.$$

Then let \mathcal{H} be the ℓ^2 space on the points x_σ, y_σ , and consider the natural representation of the Borel functions on C as multiplication operators on the elements of \mathcal{H} . Then let

$$D := \bigoplus_{\sigma \in \Sigma} \frac{1}{\|x_\sigma - y_\sigma\|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we consider the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where \mathcal{H} and D are defined as above, and \mathcal{A} is the algebra of continuous functions on C such that $[D, f]$ is bounded.

Remark 2.9. We may generalise the construction of the spectral triple by considering a finite number of ancestral pairs $\{x_i, y_i\}$, e.g., for a Sierpinski like fractal as in Figure 1, the pairs of extreme points of the three sides of the original triangle.

Theorem 2.10. *Let F be a limit fractal, $(\mathcal{A}, \mathcal{H}, D)$ the spectral triple described above, α a singular traceability exponent for $|D|^{-1}$. Then the Hausdorff-Besicovitch functional coincides, via Riesz Theorem, with the limit measure μ_α .*

Proof. Let us first assume that the ancestral pair $\{x, y\}$ is contained in V . Then the proof can be done as in Proposition 2.15 and Theorem 2.16.

Now take a generic pair $\{x', y'\}$, with $\|x' - y'\| = r'$, and the corresponding spectral triple $(\mathcal{A}, \mathcal{H}, D')$, where the Hilbert spaces are naturally identified. While the family of eigenvalues (with multiplicity) of $|D|^{-1}$ is given by

$\{r\lambda_\sigma : \sigma \in \Sigma\}$, each with multiplicity 2, the family of eigenvalues (with multiplicity) of $|D'|^{-1}$ is given by $\{r'\lambda_\sigma : \sigma \in \Sigma\}$, each with multiplicity 2. Therefore the spectral triples have the same set of traceability exponents, and if α is one of them, τ is a singular trace such that $\tau(|D|^{-\alpha}) = 1$, then $\tau'(|D'|^{-\alpha}) = 1$, with $\tau' = \left(\frac{r}{r'}\right)^\alpha \tau$.

Now let f be a Lipschitz function on C , with Lipschitz constant L . $f|D|^{-\alpha}$ is a multiplication operator, with eigenvalues (with multiplicity)

$$\{r^\alpha \lambda_\sigma^\alpha f(x_\sigma), r^\alpha \lambda_\sigma^\alpha f(y_\sigma) : \sigma \in \Sigma\}.$$

Then

$$f|D'|^{-\alpha} = \left(\frac{r'}{r}\right)^\alpha f|D|^{-\alpha} + R,$$

where R is a selfadjoint operator with eigenvalues (with multiplicity)

$$\{(r')^\alpha \lambda_\sigma^\alpha (f(x'_\sigma) - f(x_\sigma)), (r')^\alpha \lambda_\sigma^\alpha (f(y'_\sigma) - f(y_\sigma)) : \sigma \in \Sigma\}.$$

Since $|f(x'_\sigma) - f(x_\sigma)| \leq L\|x'_\sigma - x_\sigma\| = L\lambda_\sigma\|x' - x\|$, the operator $|R|$ can be majorised by a positive operator S with eigenvalues $\{LM(r')^\alpha \lambda_\sigma^{\alpha+1} : \sigma \in \Sigma\}$, each with multiplicity 2, where $M = \max(\|x' - x\|, \|y' - y\|)$. Clearly the operator S is infinitesimal w.r.t. $|D|^{-\alpha}$, hence

$$\tau'(f|D'|^{-\alpha}) = \left(\frac{r'}{r}\right)^\alpha \tau'(f|D|^{-\alpha}) + \tau'(R) = \tau(f|D|^{-\alpha}) = \int f d\mu_\alpha.$$

Since Lipschitz functions are dense, we get the thesis. \square

Remark 2.11. (i) When F is a translation fractal, and the Hausdorff dimension of the sets $F_\sigma \cap F_{\sigma'}$ is strictly lower than the Hausdorff dimension of F , $|\sigma| = |\sigma'|$, then Theorem 1.10 applies, giving

$$\mu_{\lim}(F_\sigma) = P_{|\sigma|}^{-1}.$$

(i) Let us observe that for self-similar fractals, there is only one exponent of singular traceability, namely the similarity dimension s , and the corresponding limit measure coincides with \mathcal{H}_s .

Theorem 2.12. *Let (A, D, \mathcal{H}) be the spectral triple associated with a translation fractal F where the similarities $w_{n,i}$, $i = 1, \dots, p_n$ have scaling parameter λ_n . Then the Hausdorff dimension is given by the formula*

$$d = \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k}.$$

Proof. The eigenvalues of $|D|^{-1}$ are given by r with multiplicity 2, $r\lambda_1$ with multiplicity $2p_1$, $r\lambda_1\lambda_2$ with multiplicity $2p_1p_2$, and so on. Therefore, with

$p_0 := 1$, $\lambda_0 := 1$, and Λ_n , P_n as in (1.4),

$$\begin{aligned} \text{Tr}(|D|^{-\alpha}) &= 2r^\alpha \sum_{k=0}^{\infty} \prod_{i=0}^k (p_i \lambda_i)^\alpha \\ &= 2r^\alpha \sum_{n=0}^{\infty} e^{\log P_n - \alpha \log 1/\Lambda_n} \\ &= 2r^\alpha \sum_{n=0}^{\infty} \exp \left(\log P_n \left(1 - \alpha \frac{\log 1/\Lambda_n}{\log P_n} \right) \right). \end{aligned}$$

Denote by $d := \left(\liminf_{n \rightarrow \infty} \frac{\log 1/\Lambda_n}{\log P_n} \right)^{-1}$. Then, if $\alpha > d$, we get

$$\liminf_{n \rightarrow \infty} \alpha \frac{\log 1/\Lambda_n}{\log P_n} = \frac{\alpha}{d} > 1$$

which implies that, for any sufficiently small $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that, for all $n > n_\varepsilon$,

$$1 - \alpha \frac{\log 1/\Lambda_n}{\log P_n} < -\varepsilon.$$

Since $P_n^{-\varepsilon} \leq 2^{-n\varepsilon}$, the series converges. Whereas, if $\alpha < d$, as there is a subsequence $\{n_k\}$ such that $\frac{\log 1/\Lambda_{n_k}}{\log P_{n_k}} \rightarrow d^{-1}$, we get

$$1 - \alpha \frac{\log 1/\Lambda_{n_k}}{\log P_{n_k}} \rightarrow 1 - \frac{\alpha}{d} > 0$$

and the series diverges. Therefore $d(\mathcal{A}, D, \mathcal{H}) = d$. \square

Theorem 2.13. *Let $(\mathcal{A}, \mathcal{H}, D)$ be the spectral triple associated with a translation fractal F , where the similarities $w_{n,i}$, $i = 1, \dots, p_n$ have scaling parameter λ_n . Then*

$$\begin{aligned} \underline{\delta}(\mathcal{A}, \mathcal{H}, D) &= \liminf_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}, \\ \overline{\delta}(\mathcal{A}, \mathcal{H}, D) &= \limsup_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}. \end{aligned}$$

Proof. The eigenvalues of $|D|^{-1}$ are the numbers $r\Lambda_k$, each with multiplicity $2P_k$. Therefore, the quantity $\frac{1}{h}(\log 1/\mu(e^{t+h}) - \log 1/\mu(e^t))$, may be rewritten as

$$\frac{\log 1/\Lambda_k - \log 1/\Lambda_m}{\log \left(\sum_{j=0}^k P_j - \vartheta_k P_k \right) - \log \left(\sum_{j=0}^m P_j - \vartheta'_m P_m \right)} \quad (2.2)$$

for suitable constants $\vartheta_k, \vartheta'_k$ in $[0, 1)$.

Let us observe that, since $p_i \geq 2$,

$$\begin{aligned} \log \left(\sum_{j=0}^k P_j - \vartheta_k P_k \right) - \log P_k &\leq \log \frac{\sum_{j=0}^k P_j}{P_k} \\ &= \log \left(\sum_{j=0}^k \prod_{i=j+1}^k \frac{1}{p_i} \right) \leq \log 2. \end{aligned}$$

Since the denominator goes to infinity, additive perturbations of the numerator and of the denominator by bounded sequences do not alter the \limsup , resp. \liminf , and the ratio above may be replaced by

$$\frac{\log 1/\Lambda_k - \log 1/\Lambda_m}{\log P_k - \log P_m}. \quad (2.3)$$

Finally, since the denominator $\log P_k - \log P_m$ goes to infinity if and only if $k - m \rightarrow \infty$, the thesis follows. \square

Remark 2.14. As in Connes' book, we may introduce on F the metric defined by D , namely

$$d(x, y) := \sup\{|f(x) - f(y)| : f \in \mathcal{C}(F), \|[D, f]\| \leq 1\}.$$

However it is not true, in general, that $\iota : (F, \|\cdot\|) \rightarrow (F, d)$ is a homeomorphism. For example, let F be the Sierpinski gasket, $x, y \in F$ being two vertices of the enveloping triangle. Then the metric d gives value $+\infty$ to any pair of points in F sitting on a line which is not parallel to the side x, y . Nevertheless, if we consider three ancestral pairs for F as in remark 2.9, we get the Euclidean geodesic distance in F (cf. Theorem 2.23).

2.4 A different spectral triple

In this final Subsection we construct a spectral triple on a large subclass of limit fractals. We recognise the noncommutative Hausdorff functionals as the ones arising from the limit measures, and compare the “noncommutative metric” with the Euclidean geodesic distance. In case of translation fractals, we compute the various dimensions associated to the spectral triple.

Let F be a limit fractal satisfying OSC (see Assumption 1.6). Let x_{ni} be the fixed point of w_{ni} , and assume that the set W of all x_{ni} 's is not dense in V . Fix $x_\emptyset \in V \setminus \overline{W}$, then there is $c > 0$ s.t. $\|x_\emptyset - x_{ni}\| \geq c$, for all n, i , and

$$\begin{aligned} \text{diam}(C) &\geq \|x_\emptyset - w_{ni}x_\emptyset\| \geq \left| \|x_\emptyset - x_{ni}\| - \|x_{ni} - w_{ni}x_\emptyset\| \right| \\ &= (1 - \lambda_{ni})\|x_\emptyset - x_{ni}\| \geq (1 - \overline{\lambda})c. \end{aligned} \quad (2.4)$$

Set $x_\sigma := w_\sigma x_\emptyset$, and define

$$D := \bigoplus_{\sigma \in \Sigma} \bigoplus_{i=1}^{p|\sigma|+1} \frac{1}{\|x_\sigma - x_{\sigma \cdot i}\|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

acting on

$$\mathcal{H} = \bigoplus_{\sigma \in \Sigma} \bigoplus_{i=1}^{p|\sigma|+1} \ell^2\{x_\sigma, x_{\sigma \cdot i}\}$$

In the following we shall consider the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with \mathcal{H} and D defined as above, and \mathcal{A} consisting of continuous functions f on C , acting on \mathcal{H} as

$$\begin{aligned} (f\xi)_{\sigma,i}(x_\sigma) &= f(x_\sigma)\xi_{\sigma,i}(x_\sigma) \\ (f\xi)_{\sigma,i}(x_{\sigma \cdot i}) &= f(x_{\sigma \cdot i})\xi_{\sigma,i}(x_{\sigma \cdot i}), \end{aligned}$$

and for which $[D, f]$ is bounded.

2.4.1 Measures and dimensions

For $\alpha > 0$, a singular traceability exponent of $|D|^{-1}$, set

$$\int f d\mu_\alpha^0 := \tau(f|D|^{-\alpha})$$

for any Borel function f , where τ is a singular trace such that $\tau(|D|^{-\alpha}) = 1$.

Proposition 2.15. *With the above notation, $\mu_\alpha^0(V_\sigma) = \mu_\alpha^0(C_\sigma) = c_{\sigma,\alpha}$, for any $\sigma \in \Sigma$.*

Proof. Let $\sigma, \sigma' \in \Sigma_n$, then

$$\|x_{\sigma \cdot \rho} - x_{\sigma \cdot \rho \cdot i}\| = \|w_{\sigma \cdot \rho}x_\emptyset - w_{\sigma \cdot \rho \cdot i}x_\emptyset\| = \lambda_\sigma \|w_\sigma^{-1}w_{\sigma \cdot \rho}x_\emptyset - w_\sigma^{-1}w_{\sigma \cdot \rho \cdot i}x_\emptyset\|.$$

As $w_\sigma^{-1}w_{\sigma \cdot \rho}$ is independent of σ , the operators $\lambda_\sigma^{-\alpha}\chi_{V_\sigma}|D|^{-\alpha}$ and $\lambda_{\sigma'}^{-\alpha}\chi_{V_{\sigma'}}|D|^{-\alpha}$ have the same eigenvalues (with multiplicity) up to finitely many. Therefore,

$$1 = \tau(|D|^{-\alpha}) = \sum_{\sigma' \in \Sigma_n} \tau(\chi_{V_{\sigma'}}|D|^{-\alpha}) = \sum_{\sigma' \in \Sigma_n} \frac{\lambda_{\sigma'}^\alpha}{\lambda_\sigma^\alpha} \tau(\chi_{V_\sigma}|D|^{-\alpha}),$$

so that $\mu_\alpha^0(V_\sigma) = \tau(\chi_{V_\sigma}|D|^{-\alpha}) = c_{\sigma,\alpha}$. Since $\sum_{\sigma \in \Sigma_n} c_{\sigma,\alpha} = 1$, we get $\mu_\alpha^0(C_\sigma) = c_{\sigma,\alpha}$. \square

The set function μ_α^0 is not σ -additive, however its restriction to continuous functions gives rise to the Hausdorff-Besicovitch functional on $\mathcal{C}(F) = \overline{\mathcal{A}}$. The following holds:

Theorem 2.16. *Let α be a traceability exponent. Then the measure associated with the Hausdorff-Besicovitch functional via the Riesz Theorem, coincides with the limit measure μ_α . In particular it does not depend either on x_\emptyset or on the generalised limit.*

Proof. Let us show that the regularization $\bar{\mu}_\alpha$ of μ_α^0 satisfies the inequalities (1.3). Indeed

$$\bar{\mu}_\alpha(V_J) \leq \mu_\alpha^0(V_J) \leq \sum_{\sigma \in J} \mu_\alpha^0(C_\sigma) = \sum_{\sigma \in J} c_{\sigma, \alpha}.$$

Moreover,

$$\bar{\mu}_\alpha(C_J) \geq \mu_\alpha^0(C_J) \geq \sum_{\sigma \in J} \mu_\alpha^0(V_\sigma) = \sum_{\sigma \in J} c_{\sigma, \alpha}.$$

Then, by the uniqueness proved in Theorem 1.7, $\bar{\mu}_\alpha$ coincides with μ_α . \square

Theorem 2.17. *Let $(\mathcal{A}, D, \mathcal{H})$ be the spectral triple associated with a translation fractal F , where the similarities $w_{n,i}$, $i = 1, \dots, p_n$ have scaling parameter λ_n , and $\sup_n p_n < +\infty$. Then*

$$\begin{aligned} d(\mathcal{A}, \mathcal{H}, D) &= \limsup_n \frac{\sum_1^n \log p_k}{\sum_1^n \log 1/\lambda_k} \\ \underline{\delta}(\mathcal{A}, \mathcal{H}, D) &= \liminf_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}, \\ \bar{\delta}(\mathcal{A}, \mathcal{H}, D) &= \limsup_{n,k} \frac{\sum_{j=n}^{n+k} \log p_j}{\sum_{j=n}^{n+k} \log 1/\lambda_j}. \end{aligned}$$

Proof. The eigenvalues of $|D|^{-1}$ are the numbers $\bar{\Lambda}_{k,i} := \Lambda_k \|x_\emptyset - w_{k+1,i} x_\emptyset\|$, each with multiplicity $2P_k$. It follows from (2.4) that there are $0 < c_1 < c_2$ s.t. $c_1 \Lambda_k \leq \bar{\Lambda}_{k,i} \leq c_2 \Lambda_k$, for all $k \in \mathbb{N}$. Therefore

$$Tr(|D|^{-\alpha}) = \sum_{k \in \mathbb{N}} P_k \sum_{i=1}^{p_{k+1}} \bar{\Lambda}_{k,i}^\alpha \asymp \sum_k P_k \Lambda_k^\alpha,$$

and the first equality follows as in Theorem 2.12. As for the others, the same computation in the proof of Theorem 2.13 can be performed. \square

From Theorems 1.14, 2.17 we get

Corollary 2.18. *Let F be a translation (limit) fractal, μ its limit measure, $(\mathcal{A}, \mathcal{H}, D)$ its associated spectral triple. Then, for any $x \in F$,*

$$\begin{aligned} d(\mathcal{A}, \mathcal{H}, D) &= \bar{d}_\mu(x), \\ \underline{\delta}(\mathcal{A}, \mathcal{H}, D) &= \underline{\delta}_\mu(x), \\ \bar{\delta}(\mathcal{A}, \mathcal{H}, D) &= \bar{\delta}_\mu(x). \end{aligned}$$

2.4.2 Metrics

Let us now introduce on F the metric defined by D as in Connes' book, namely

$$d(x, y) := \sup\{|f(x) - f(y)| : f \in \mathcal{C}(C), \|[D, f]\| \leq 1\}.$$

Lemma 2.19. $\|x - y\| \leq d(x, y)$, for all $x, y \in F$.

Proof. Let f be a Lipschitz function on C with Lipschitz constant ≤ 1 . Then $\|[D, f]\| \leq 1$, therefore $\|x - y\| \leq d(x, y)$. \square

Let us observe first that this distance and the Euclidean distance induce the same topology.

Lemma 2.20. (i) Let $\sigma \in \Sigma_n$, $x \in F_\sigma$. Then $d(x, x_\sigma) \leq \frac{\text{diam}(C)}{1-\lambda} \lambda_\sigma$.

(ii) Let $\sigma, \sigma' \in \Sigma_n$ be s.t. $F_\sigma \cap F_{\sigma'} \neq \emptyset$. Then $d(x_\sigma, x_{\sigma'}) \leq \frac{\text{diam}(C)}{1-\lambda} (\lambda_\sigma + \lambda_{\sigma'})$.

Proof. (i). There is $\rho \in \Sigma_\infty$ s.t. $x = x_\rho$ and the n -th truncation ρ^n of ρ is equal to σ , i.e., $\rho^n(i) = \sigma(i)$, $i = 1, \dots, n$. Then

$$\begin{aligned} d(x, x_\sigma) &\leq \sum_{k=|\sigma|}^{\infty} d(x_{\rho^k}, x_{\rho^{k+1}}) \\ &\leq \sum_{k=|\sigma|}^{\infty} \lambda_{\rho^k} \text{diam}(C) \leq \frac{\text{diam}(C)}{1-\lambda} \lambda_\sigma. \end{aligned}$$

(ii) Let $x \in F_\sigma \cap F_{\sigma'}$. Then $d(x_\sigma, x_{\sigma'}) \leq d(x, x_\sigma) + d(x, x_{\sigma'}) \leq \frac{\text{diam}(C)}{1-\lambda} (\lambda_\sigma + \lambda_{\sigma'})$. \square

Theorem 2.21. Let F be a limit fractal. Then $\iota : (F, \|\cdot\|) \rightarrow (F, d)$ is a homeomorphism.

Proof. Assume ι is discontinuous in $x \in F$, so that there are $c > 0$, $\{x_n\} \subset F$ s.t. $\|x_n - x\| \rightarrow 0$, and $d(x_n, x) \geq c$, $n \in \mathbb{N}$. From Lemma 2.20 we obtain $k \in \mathbb{N}$ s.t., for $\sigma \in \Sigma_k$, we have $\text{diam}(F_\sigma) < \frac{c}{2}$. As $\cup_{\sigma \in \Sigma_k} F_\sigma = F$, there is $\sigma \in \Sigma_k$ s.t. $A := \{n \in \mathbb{N} : x_n \in F_\sigma\}$ is infinite, hence $x \in F_\sigma$. Therefore $c \leq d(x_n, x) \leq \text{diam}(F_\sigma) < \frac{c}{2}$, which is absurd. \square

The map ι is not bi-Lipschitz in general. This is true however when the Euclidean distance and the Euclidean geodesic distance in F are bi-Lipschitz.

Lemma 2.22. Let $A \subset \mathbb{R}^N$ be a bounded open set, $\rho \geq 1$. Then there are $k \in \mathbb{N}$, $c > 0$, such that, given similarities $\varphi_1, \dots, \varphi_k$ such that $\varphi_i(A) \cap \varphi_j(A) = \emptyset$, $i \neq j$, with similarity parameters $\lambda_1 \leq \dots \leq \lambda_k$ satisfying $\lambda_k \leq \rho \lambda_1$, and a rectifiable curve γ such that $\gamma \cap \varphi_i(\overline{A}) \neq \emptyset$, $\forall i = 1, \dots, k$, we have $\ell(\gamma) > c \lambda_1$.

Proof. Possibly rescaling, it is not restrictive to assume that $\lambda_1 = 1$. Let $k \in \mathbb{N}$ be s.t. the infimum length of a rectifiable curve intersecting the closure of k dilated copies of A with disjoint interior is 0. Then, for any $n \in \mathbb{N}$, we get sequences of dilations $\varphi_{1n}, \dots, \varphi_{kn}$ such that $\varphi_{in}(A) \cap \varphi_{jn}(A) = \emptyset$, and a rectifiable curve γ_n with $\ell(\gamma_n) < \frac{1}{n}$ such that $\gamma_n \cap \varphi_{in}(\overline{A}) \neq \emptyset$, $i = 1, \dots, k$, $n \in \mathbb{N}$. Clearly it is not restrictive to assume that all curves γ_n start from the same point x_0 , which implies that all the curves and copies of A are contained

in some compact set. By the assumptions above, all dilations lie in a compact set. Hence, possibly passing to a subsequence, we may assume that, for any $i = 1, \dots, k$, φ_{in} converges to a dilation φ_i of \mathbb{R}^N and $\gamma_n \rightarrow \gamma$ in the Hausdorff topology. Let us remark that in this way $\overline{A_{in}} \rightarrow \overline{A_i}$ in the Hausdorff topology, where $A_i := \varphi_i(A)$, $A_{in} := \varphi_{in}(A)$, $i = 1, \dots, k$, $n \in \mathbb{N}$. As $\text{diam}(\gamma) = 0$, γ consists of a single point, indeed the point x_0 . Moreover $\gamma \cap \overline{A_i} \neq \emptyset$, for all i , i.e. $x_0 \in \bigcap_{i=1}^k \overline{A_i}$.

We claim that A_1, \dots, A_k are disjoint. By contradiction, assume that $A_i \cap A_j$ is not empty, namely that there exist points $x_i, x_j \in A$ with $\varphi_i(x_i) = \varphi_j(x_j)$, and let $r > 0$ be s.t. $B(x_i, r), B(x_j, r) \subset A$. Since the φ_i 's are dilations, this implies that $B(\varphi_i(x_i), r) \subset A_i$, and the same for j . We have that $\varphi_{in}(x_i)$ and $\varphi_{jn}(x_j)$ converge to the same point, hence, for a sufficiently large n , $\|\varphi_{in}(x_i) - \varphi_{jn}(x_j)\| < r$, so that $A_{in} \cap A_{jn} \neq \emptyset$, which is absurd. Since $x_0 \in \bigcap_{i=1}^k \overline{A_i}$, so that $A_i \subset B(x_0, \rho \text{diam}(A))$, we obtain

$$\omega_N \rho^N (\text{diam}(A))^N = \text{vol}(B(x_0, \rho \text{diam}(A))) \geq \sum_{i=1}^k \text{vol}(A_i) \geq k \text{vol}(A),$$

with ω_N the volume of the unit ball in \mathbb{R}^N . Therefore, if we take the constant k greater than $\frac{\omega_N \rho^N (\text{diam}(A))^N}{\text{vol}(A)}$, the infimum length of a rectifiable curve intersecting k dilated copies of A with disjoint interior cannot be 0. \square

Theorem 2.23. *Let F be a limit fractal in \mathbb{R}^N , d_{geo} the Euclidean geodesic distance in F , namely the distance defined in terms of rectifiable curves contained in the fractal (when they exist). Assume $\underline{\lambda} := \inf_{n,i} \lambda_{ni} > 0$. Then there is $c \geq 1$ s.t. $\|x - y\| \leq d(x, y) \leq c d_{geo}(x, y)$, $x, y \in F$.*

Proof. The first inequality was proved above, let us prove the second. For any $\varepsilon > 0$, consider the set $\Sigma(\varepsilon)$ consisting of the multi-indices σ such that $\lambda_\sigma \leq \varepsilon$ but $\lambda_{\sigma^{n-1}} > \varepsilon$, where $n = |\sigma|$ and σ^k denotes the k -th truncation of σ (as in the proof of Lemma 2.20). Then $\lambda_\sigma = \lambda_{n, \sigma(n)} \lambda_{\sigma^{n-1}} > \underline{\lambda} \varepsilon$. It is clear that $\bigcup_{\sigma \in \Sigma(\varepsilon)} F_\sigma = F$, and $V_\sigma \cap V_{\sigma'} = \emptyset$ if $\sigma \neq \sigma'$ are in $\Sigma(\varepsilon)$.

Now let $x, y \in F$, $\varepsilon > 0$. Choose a rectifiable curve γ in F connecting x and y , with $\ell(\gamma) < 2d_{geo}(x, y)$, and let $\sigma_1, \dots, \sigma_k$ be the elements of $\Sigma(\varepsilon)$ such that $C_{\sigma_i} \cap \gamma \neq \emptyset$, ordered in such a way that $x \in C_{\sigma_1}$, $y \in C_{\sigma_k}$, and $C_{\sigma_i} \cap C_{\sigma_{i+1}} \neq \emptyset$, $i = 1, \dots, k-1$.

By Lemma 2.20 we get

$$\begin{aligned} d(x, y) &\leq d(x, x_{\sigma_1}) + \sum_{i=1}^{k-1} d(x_{\sigma_i}, x_{\sigma_{i+1}}) + d(x_{\sigma_k}, y) \\ &\leq \frac{2 \text{diam}(C)}{1 - \bar{\lambda}} \sum_{i=1}^k \lambda_{\sigma_i} \leq \frac{2 \text{diam}(C) k \varepsilon}{1 - \bar{\lambda}}. \end{aligned}$$

Let us notice that $k \rightarrow \infty$, when $\varepsilon \rightarrow 0$.

Now let $k_V \in \mathbb{N}$, $c_V > 0$ be the constants associated to V and to $1/\underline{\lambda}$ in Lemma 2.22. Then

$$\begin{aligned} \ell(\gamma) &\geq [k/k_V] c_V \underline{\lambda} \varepsilon \\ &\geq \frac{c_V \underline{\lambda} (1 - \overline{\lambda})}{2 \operatorname{diam}(C)} \frac{[k/k_V]}{k} d(x, y). \end{aligned}$$

Passing to the limit for $\varepsilon \rightarrow 0$, we get $d(x, y) \leq \frac{4k_V \operatorname{diam}(C)}{c_V \underline{\lambda} (1 - \overline{\lambda})} d_{\text{geo}}(x, y)$, i.e. the thesis. \square

Remark 2.24. For many fractals, as Sierpinski gasket and carpet, Vicsek, Lindstrom snowflake etc., d_{geo} and the Euclidean distance are biLipschitz, hence also d and the Euclidean distance are.

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